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EXISTENCE AND STABILITY OF A TRAVELING WAVE SOLUTION ON A 3-COMPONENT REACTION-DIFFUSION MODEL IN COMBUSTION

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1. Introduction

It is shown in [8] that thin solid, for an example, paper, cellulose dialysis bags and polyethylene sheets, burning against oxidizing wind develops finger-like patterns or fingering patterns. The oxidizing gas is supplied in a uniform laminar flow, opposite to the directions of the front propagation and they control the flow velocity of oxygen, denoted by V. When V is decreased below a critical value, the smooth front develops a structure which marks the onset of instability. As V is decreased further, the peaks are separated by cusp-like minima and a fingering pattern is formed. In addition, thin solid is stretched out straight onto the bottom plate and they also control the adjustable vertical gap, denoted by a parameter h, between top and bottom plates. We remark here that fingering patterns occur for small h, which implies that such patterns appear in the absence of natural convection. Similar phenomena have been also observed in a micro-gravity experiment in space (see [5]).

To investigate these phenomena, a reaction-diffusion model (RD) was proposed in [2]. We carried out numerical simulations, reproducing similar results to the experiment described above. If the effect of the flow (denoted by λ in (RD)) is strong, a flame front is smooth. Decreasing λ raises the destabilization of the smooth flame front. Eventually, fingering pattern occurs in small $\lambda > 0$.

Our model (RD) is represented as follows:

$$\text{(RD)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = Le\Delta u + \lambda' \dfrac{\partial u}{\partial x} + \gamma k(u)vw - au, \\ \dfrac{\partial v}{\partial t} = -k(u)vw, \\ \dfrac{\partial w}{\partial t} = \Delta w + \lambda \dfrac{\partial w}{\partial x} - k(u)vw, \end{cases} \qquad (x,y) \in (-\infty,\infty) \times \Omega, t > 0,$$

where the constants Le, called Lewis number, γ and a are positive constants, λ and λ' are nonnegative constants, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\Delta = \partial^2/\partial x^2 + \sum_{i=1}^n \partial^2/\partial y_i^2$ is Laplacian as usual. The nonlinear term k is defined by

$$k(u) = \begin{cases} A\exp(-B/(u-\theta)), & u > \theta, \\ 0, & 0 \leq u \leq \theta \end{cases}$$

for some constants $A, B > 0$ and $\theta \geq 0$. This function k and θ are called *Arrhenius kinetics* and *ignition temperature* in combustion. Note that we considered a general setting for the nonlinear function k in [2] and [3].

We suppose that

$$\lim_{|x|\to\infty} u(x,y,t) = 0, \quad \lim_{x\to\infty} w(x,y,t) = w_r > 0, \quad \lim_{x\to-\infty} w(x,y,t) = w_l \geq 0$$

for any $y \in \Omega$ and $t > 0$, where w_r and w_l are constants and $w_r > w_l$. We also suppose that u and w satisfy

$$\frac{\partial u}{\partial \nu}(x,y,t) = 0, \quad \frac{\partial w}{\partial \nu}(x,y,t) = 0$$

for $x \in (-\infty,\infty)$, $y \in \partial\Omega$ and $t > 0$, where ν is the unit exterior normal vector on $\partial\Omega$. We suppose that initial functions satisfy

$$u(x,y,0) = u_0(x,y) \geq 0, \quad v(x,y,0) = v_0(x,y) \geq 0, \quad w(x,y,0) = w_0(x,y) \geq 0,$$

and

(1.1) $$w_0(+\infty, y) = w_r, \quad w_0(-\infty, y) = w_l.$$

In numerical simulations, a smooth flame front is observed in (RD) if λ is sufficiently large, which implies that (RD) has a stable traveling wave solution independent of y-variable. Our first aim in this paper is to construct a stable traveling wave solution in the case that λ is large. The second aim will be described after the statement of Theorem 3.

Now we describe main results and how to prove the existence and stability of a traveling wave solution of (RD). We formally take the limit of $\lambda \to \infty$ in (RD) so that $\partial w/\partial x = 0$ holds. Then, from the boundary condition of w, we obtain $w \equiv w_r$ and (RD) is reduced to

(1.2) $$\begin{cases} \dfrac{\partial u}{\partial t} = Le\Delta u + \lambda' \dfrac{\partial u}{\partial x} + \gamma k(u) v w_r - au, \\ \dfrac{\partial v}{\partial t} = -k(u) v w_r \end{cases} \qquad (x, y) \in (-\infty, \infty) \times \Omega, t > 0$$

with the boundary condition

$$\lim_{|x| \to \infty} u(x, y, t) = 0, \quad y \in \Omega, t > 0,$$

$$\frac{\partial u}{\partial \nu}(x, y, t) = 0, \quad x \in (-\infty, \infty), y \in \partial\Omega, t > 0.$$

Hence a solution of (RD) approaches that of (1.2).

Theorem 1. *Let $(u^\lambda, v^\lambda, w^\lambda)$ be a solution of (RD) with an initial function $(u_0^\lambda, v_0^\lambda, w_0^\lambda)$ depending on λ and (u, v) be a solution of (1.2) with an initial function (u_0, v_0). Suppose that $(u_0^\lambda, v_0^\lambda)$ and (u_0, v_0) belong to $D(L_u^\alpha) \times C^\kappa((-\infty, \infty) \times \Omega)$ and satisfy*

(1.3) $$\|u_0^\lambda - u_0\|_\alpha \to 0, \quad \|v_0^\lambda - v_0\|_{L^\infty((-\infty,\infty)\times\Omega)} \to 0$$

as $\lambda \to \infty$. Here L_u^α is a fractional power of $L_u \equiv -Le\Delta - \lambda'\partial/\partial x + a$ with the domain $D(L_u^\alpha)$ endowed by $\|\cdot\|_\alpha \equiv \|\cdot\|_{L^p((-\infty,\infty)\times\Omega)} + \|L_u^\alpha \cdot\|_{L^p((-\infty,\infty)\times\Omega)}$ for $1/2 < \alpha < 1$ and $n + 1 < p < \infty$ (see [6]), and $C^\kappa((-\infty, \infty) \times \Omega)$ is a Hölder space with the exponent $0 < \kappa < 1$. In addition, assume $w_0^\lambda - \eta \in D(L_z^\alpha)$, where a monotonically increasing function $\eta \in C^2(-\infty, \infty)$ satisfy

$$\eta(x) = \begin{cases} w_r, & x \ge 1, \\ w_l, & x \le 0, \end{cases}$$

and L_z^α is a fractional power of $L_z \equiv -\Delta - \lambda\partial/\partial x$. Then, for any $\delta, T > 0$ and $R \in (-\infty, \infty)$,

(1.4) $$\begin{aligned} &\sup_{0<t<T} (\|u^\lambda(t) - u(t)\|_\alpha + \|v^\lambda(t) - v(t)\|_{L^\infty((-\infty,\infty)\times\Omega)}) \to 0, \\ &\sup_{\delta<t<T} \|w^\lambda(t) - w_r\|_{L^\infty((R,\infty)\times\Omega)} \to 0 \end{aligned}$$

as $\lambda \to \infty$.

From this result, a traveling wave solution of (RD) may approach that of (1.2). In order to achieve our goal, we introduce a new parameter $\varepsilon > 0$ and construct a solution of

(1.5) $$\begin{cases} -\varepsilon c u' = \varepsilon^2 u'' + \varepsilon\lambda' u' + \gamma k(u) v w_r - au, \\ -cv' = -k(u) v w_r \end{cases}$$

with boundary conditions

(1.6) $$u(\pm\infty) = 0, \quad v(+\infty) = v_r,$$

where c is called wave speed of a traveling wave solution. We derived (1.5) from (1.2) by putting $Le \to \varepsilon$, $\gamma \to \gamma/\varepsilon$, and $a \to a/\varepsilon$. Although this problem is easier than (1.8) and (1.9) below, it is still difficult to verify the existence of a traveling wave solution without any technical assumptions for parameters. If we use the smal parameter ε, we can apply perturbation theory to our problem and construct a traveling wave solution. By this method we also see how the traveling wave solution obtained in the following theorem behaves as $\varepsilon \to 0$, and that it is stable in (1.2). This is why we introduced the small parameter $\varepsilon > 0$ above.

Theorem 2 ([3]). *Suppose that there is $\underline{v}$ such that for any $\underline{v} < v$, it holds that*

$$\int_0^{u_1(\underline{v})} (\gamma k(u)\underline{v}w_r - au)du = 0,$$

where $u_1(v)$ denotes the maximum of the three zeroes of $\gamma k(u)vw_r - au$. Then, there are positive constants $\overline{v}$ and $\lambda'(v_r)$ such that if $\underline{v} < v_r < \overline{v}$, $0 \leq \lambda' < \lambda'(v_r)$, and $\varepsilon > 0$ is sufficiently small, the system (1.5) with (1.6) has a solution, denoted by (u, v, c). In addition, the associated eigenvalue problem

$$\text{(1.7)} \qquad \begin{cases} \varepsilon\mu\phi = \varepsilon^2\phi'' + \varepsilon(c+\lambda')\phi' + \gamma k'(u)vw_r\phi + \gamma k(u)w_r\psi - a\phi, \\ \mu\psi = c\psi' - k'(u)vw_r\phi - k(u)\psi \end{cases}$$

has a unique solution $(\phi, \psi, \mu) = (u', v', 0)$ in $H^2_\kappa(\mathbb{R}) \times H^1_\kappa(\mathbb{R}) \times \Lambda_\delta$ for small $\kappa > 0$, where $H^1_\kappa(\mathbb{R})$ and $H^2_\kappa(\mathbb{R})$ are weighted Sobolev spaces, and Λ_δ is a closed subset in $\mathbb{C}$ for small $\delta > 0$ defined later. The two small parameters κ and δ are supposed to be independent of ε. Furthermore the algebraic multiplicity of $\mu = 0$ is 1 in (1.7).

A traveling wave solution is (linearly) stable if the eigenvalue problem does not have an eigenvalue $\mu \in \Lambda_\delta$ except for $\mu = 0$, and the algebraic multiplicity of $\mu = 0$ is 1. Note that (u', v') is a solution of (1.7) for $\mu = 0$. Since $k(0) = 0$ and $k'(0) = 0$, the essential spectra come to the imaginary axis if we consider the above problem in a usual Lebesgue space or continuous function's space (see Section 5 in [1]). In order to avoid the essential spectra of (1.10), it is necessary to introduce weighted functional spaces. We define a functional space $L^2_\kappa(\mathbb{R})$ by

$$L^2_\kappa(\mathbb{R}) = \left\{ \varphi \in L^1_{loc}(\mathbb{R}) \;\middle|\; \|\varphi\|_{L^2_\kappa} \equiv \left(\int_{-\infty}^{\infty} |\varphi(z)|^2 e^{2\kappa z} dz \right)^{1/2} < \infty \right\}.$$

Sobolev spaces $H^1_\kappa(\mathbb{R})$ and $H^2_\kappa(\mathbb{R})$ with the weight function $e^{\kappa z}$ are defined as $L^2_\kappa(\mathbb{R})$ analogously. If we assume that the eigenfunction belongs to the weighted space, the eigenvalue problem (1.10) does not have essential spectra in $\mu \in \Lambda_\delta$ for a small $\delta > 0$ Hence it is sufficient to consider only spectra with a finite multiplicity (namely, eigenvalues), where Λ_δ is defined by

$$\Lambda_\delta = \{\mu \in \mathbb{C} \mid \mathrm{Re}\mu \geq -\delta\}$$

and $\mathrm{Re}\mu$ is the real part of μ. Although we only consider the linear stability in this paper, it may imply the usual stability.

From Theorems 1 and 2, we can easily obtain a stable traveling wave solution in (RD) as a perturbed solution of (1.5) and (1.6). However, we cannot obtain a traveling wave solution in (RD) by only Theorems 1 and 2 because Theorem 1 determines the behavior of solutions in (RD) and (1.2) in local time. We have to give a rigorous proof in order to establish the existence of a traveling wave solution in (RD).

We follow the argument above and use the small parameter ε. Our problem is given by

$$\text{(1.8)} \qquad \begin{cases} -\varepsilon cu' = \varepsilon^2 u'' + \varepsilon\lambda' u' + \gamma k(u)vw - au, \\ -cv' = -k(u)vw, \\ -cw' = w'' + \lambda w' - k(u)vw, \end{cases}$$

and boundary conditions

$$\text{(1.9)} \qquad u(\pm\infty) = 0, \quad v(+\infty) = v_r > 0, \quad w(+\infty) = w_r,$$

where the spatial coordinate z is given by $z = x - ct$.

Theorem 3. *Under the same conditions as in Theorem 2, if λ is sufficiently large, there is a traveling wave solution, denoted by (u, v, w, c) of (1.8) and (1.9). In addition, the associated eigenvalue problem*

$$\text{(1.10)} \qquad \begin{cases} \varepsilon\mu\phi = \varepsilon^2\phi'' + \varepsilon(c+\lambda')\phi' + \gamma k'(u)vw\phi + \gamma k(u)w\psi + \gamma k(u)v\eta - a\phi, \\ \mu\psi = c\psi' - k'(u)vw\phi - k(u)w\psi - k(u)v\eta, \\ \mu\eta = \eta'' + (c+\lambda)\eta' - k'(u)vw\phi - k(u)w\psi - k(u)v\eta \end{cases}$$

has a unique solution $(\phi, \psi, \eta, \mu) = (u', v', w', 0)$ *in* $H^2_\kappa(\mathbb{R}) \times H^1_\kappa(\mathbb{R}) \times C_\kappa(\mathbb{R}) \times \Lambda_\delta$, *where* $C_\kappa(\mathbb{R})$ *is defined by*

$$C_\kappa(\mathbb{R}) = \{\eta \in C(\mathbb{R}) \mid \sup_{-\infty<z<\infty} |\eta(z)|e^{\kappa z} < \infty\}.$$

Furthermore the algebraic multiplicity of $\mu = 0$ *is* 1.

So far we have been investigating a traveling wave solution which represents flame uniformly burning against oxidizing wind. By numerical calculation we observe another type of solutions in (RD), "reflection of traveling wave solutions" (see Figure 1, [4]). Our second aim in this paper is to consider the reflection phenomena in (RD). Actually, reflection cannot be seen in the case that λ is large. In the above we only consider a traveling wave solution under the condition that λ is sufficiently large, which cannot be applied to reflection phenomena. Then we construct a solution of (1.8) with λ fixed again.

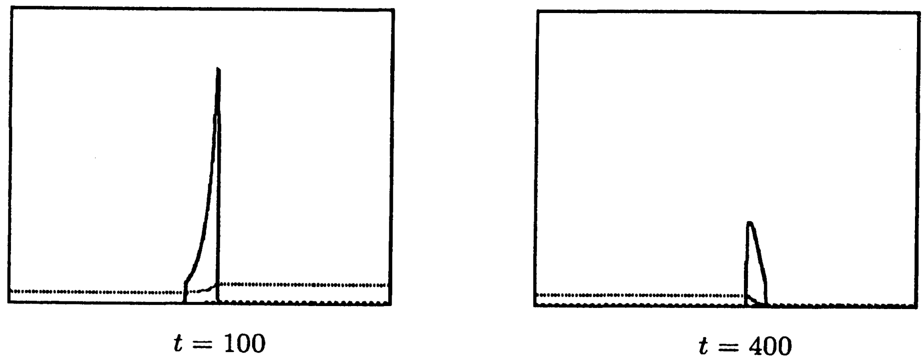

FIGURE 1. Reflection of a traveling wave solution. In this figure, three lines (one solid line and two dotted lines) represent the functions T, P, and W, respectively. This numerical calculation was done in a finite interval. The traveling wave solution initially goes to right (the left figure). After it hits the boundary, a different traveling wave solution arises (the right figure).

Theorem 4. *Fix* λ. *Under the same conditions as in Theorem 2, there is a traveling wave solution of (1.8) and (1.9).*

We also consider other traveling wave solution in (RD) in the opposite direction of the previous traveling wave solution and study

$$(1.11) \qquad \begin{cases} \varepsilon c u' = \varepsilon^2 u'' + \varepsilon \lambda' u' + \gamma k(u) v w - a u, \\ c v' = -k(u) v w, \\ c w' = w'' + \lambda w' - k(u) v w, \end{cases}$$

and boundary conditions

$$(1.12) \qquad u(\pm\infty) = 0, \quad v(-\infty) = v_r, \quad w(+\infty) = w_r.$$

Theorem 5. *Fix* λ *independent of* ε. *Under the same conditions as in Theorem 2, there is a traveling wave solution of (1.8) and (1.9).*

Here we remark a related result on the existence of a traveling wave solution of (1.5). This is the work of Roques [7]. In this work, the author proved the existence of a traveling wave solution in a combustion model with an ignition temperature (i.e. $\theta > 0$ in the definition of $k(u)$) without using any singular perturbation theory. This result implies that (1.5) has only two traveling wave solutions with different wave speeds. However, this work does not contain the case where $k(u)$ is not of ignition type, namely, $k(u) > 0$ for $u > 0$. In addition, the stability of those traveling wave solutions is unclear although it may be believed that a traveling wave solution with a faster wave speed is stable and a traveling wave solution with a slower wave speed is unstable in general. On the other hand, we prove the existence of a traveling wave solution even in the case of $\theta = 0$. Furthermore, we also show the stability of that traveling wave solution by using a singular perturbation theory.

This paper is organized as follows. In what follows we only give an outline of the proof for Theorems 4 and 5. In the proof we apply singular perturbation theory. We formally construct solutions, called outer and inner solutions.

2. Construction of a traveling wave solution in (1.8) and (1.11)

In this section we construct a formal solution of (1.8) and (1.11). We set $z \to -z$ and rewrite (1.8) into

$$\begin{cases} \varepsilon c u' = \varepsilon^2 u'' - \varepsilon\lambda' u' + \gamma k(u) v w - a u, \\ c v' = -k(u) v w, \\ c w' = w'' - \lambda w' - k(u) v w, \end{cases} \tag{2.1}$$

and boundary conditions

$$u(\pm\infty) = 0, \quad v(-\infty) = v_r, \quad w(-\infty) = w_r. \tag{2.2}$$

We first construct outer and inner solutions of this problem. We divide $(-\infty, \infty)$ into three parts

$$I_1 = (-\infty, 0), \quad I_2 = (0, \tau), \quad I_3 = (\tau, \infty).$$

The width of the second interval is a parameter denoted by τ, which is determined later. From the second and third equations of (2.1), we have

$$w'' - (c + \lambda) w' = k(u) v w = -c v'.$$

By integrating $(-\infty, z)$, it holds that

$$w' - (c + \lambda)(w - w_r) = -c(v - v_r).$$

We treat this equation instead of the third equation of (2.1). Finally, we consider on each intervals

$$\begin{cases} \varepsilon^2 u^{(1)''} - \varepsilon(c + \lambda') u^{(1)'} + \gamma k(u^{(1)}) v^{(1)} w^{(1)} - a u^{(1)} = 0, & z \in I_1, \\ c v^{(1)'} + k(u^{(1)}) v^{(1)} w^{(1)} = 0, & z \in I_1, \\ w^{(1)'} - (c + \lambda)(w^{(1)} - w_r) = -c(v^{(1)} - v_r), & z \in I_1, \end{cases} \tag{2.3}$$

$$\begin{cases} \varepsilon^2 u^{(2)''} - \varepsilon(c + \lambda') u^{(2)'} + \gamma k(u^{(2)}) v^{(2)} w^{(2)} - a u^{(2)} = 0, & z \in I_2, \\ c v^{(2)'} + k(u^{(2)}) v^{(2)} w^{(2)} = 0, & z \in I_2, \\ w^{(2)'} - (c + \lambda)(w^{(2)} - w_r) = -c(v^{(2)} - v_r), & z \in I_2, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \varepsilon^2 u^{(3)''} - \varepsilon(c + \lambda') u^{(3)'} + \gamma k(u^{(3)}) v^{(3)} w^{(3)} - a u^{(3)} = 0, & z \in I_3, \\ c v^{(3)'} + k(u^{(3)}) v^{(3)} w^{(3)} = 0, & z \in I_3, \\ w^{(3)'} - (c + \lambda)(w^{(3)} - w_r) = -c(v^{(3)} - v_r), & z \in I_3. \end{cases} \tag{2.5}$$

Also, we construct a formal solution of (1.11) by dividing $(-\infty, \infty)$ into three parts

$$I_1 = (-\infty, 0), \quad I_2 = (0, \tau), \quad I_3 = (\tau, \infty).$$

Since our traveling wave solution is expected to be bounded, the function w must converge to a constant, denoted by w_l, as $z \to -\infty$ if exists. Since w_l represents the density of oxygen in the direction where flame proceeds, w_l must be nonnegative and less than w_r. By the same argument as above, we replace the third equation of (1.11) into a first-order differential equation and consider on each intervals

$$\text{(2.6)}\qquad \begin{cases} \varepsilon^2 u^{(1)''} + \varepsilon(\lambda' - c)u^{(1)'} + \gamma k(u^{(1)})v^{(1)}w^{(1)} - au^{(1)} = 0, & z \in I_1, \\ cv^{(1)'} + k(u^{(1)})v^{(1)}w^{(1)} = 0, & z \in I_1, \\ w^{(1)'} + (\lambda - c)(w^{(1)} - w_l) = -c(v^{(1)} - v_r), & z \in I_1, \end{cases}$$

$$\text{(2.7)}\qquad \begin{cases} \varepsilon^2 u^{(2)''} + \varepsilon(\lambda' - c)u^{(2)'} + \gamma k(u^{(2)})v^{(2)}w^{(2)} - au^{(2)} = 0, & z \in I_2, \\ cv^{(2)'} + k(u^{(2)})v^{(2)}w^{(2)} = 0, & z \in I_2, \\ w^{(2)'} + (\lambda - c)(w^{(2)} - w_l) = -c(v^{(2)} - v_r), & z \in I_2, \end{cases}$$

and

$$\text{(2.8)}\qquad \begin{cases} \varepsilon^2 u^{(3)''} - \varepsilon(\lambda' - c)u^{(3)'} + \gamma k(u^{(3)})v^{(3)}w^{(3)} - au^{(3)} = 0, & z \in I_3, \\ cv^{(3)'} - k(u^{(3)})v^{(3)}w^{(3)} = 0, & z \in I_3, \\ w^{(3)'} + (\lambda - c)(w^{(3)} - w_l) = -c(v^{(3)} - v_r), & z \in I_3. \end{cases}$$

The nonnegative constant w_l will be determined later.

2.1. **The lowest order approximation of (2.1).** We first construct *outer solutions*. By putting $\varepsilon = 0$ in (2.3), we formally get

$$\begin{cases} \gamma k(U_0^{(1)})V_0^{(1)}W_0^{(1)} - aU_0^{(1)} = 0, & z \in (-\infty, 0), \\ cV_0^{(1)'} + K(U_0^{(1)})V_0^{(1)}W_0^{(1)} = 0, & z \in (-\infty, 0), \\ W_0^{(1)'} - (c + \lambda)(W_0^{(1)} - w_r) = -c(V_0^{(1)} - v_r), & z \in (-\infty, 0), \\ V_0^{(1)}(-\infty) = v_r, \quad W_0^{(1)}(-\infty) = w_r. \end{cases}$$

From the first and second equations it holds that $U_0^{(1)}(z) = 0$ and $V_0^{(1)}(z) = v_r$. Then $W_0^{(1)}(z)$ is given by

$$W_0^{(1)}(z) = w_r - Ae^{(c+\lambda)z}$$

for a constant A determined later.

Next, by putting $\varepsilon = 0$ in (2.4), we formally get

$$\begin{cases} \gamma k(U_0^{(2)})V_0^{(2)}W_0^{(2)} - aU_0^{(2)} = 0, & z \in (0, \tau), \\ cV_0^{(2)'} + k(U_0^{(2)})V_0^{(2)}W_0^{(2)} = 0, & z \in (0, \tau), \\ W_0^{(2)'} - (c + \lambda)(W_0^{(2)} - w_r) = -c(V_0^{(2)} - v_r), & z \in (0, \tau), \\ V_0^{(2)}(0) = V_0^{(1)}(0), \quad W_0^{(2)}(0) = W_0^{(1)}(0). \end{cases}$$

Let $p = h_+(q)$ be a unique positive solution of $\gamma k(p)q - aq = 0$. Then the first equation can be solved with respect to $U_0^{(2)}$ such as $U_0^{(2)}(z) = h_+(V_0^{(2)}(z)W_0^{(2)}(z))$. Substituting it into the second equation, we have

$$\begin{cases} cV_0^{(2)'} = -k(h_+(V_0^{(2)}W_0^{(2)}))V_0^{(2)}W_0^{(2)}, & z \in (0, \tau), \\ W_0^{(2)'} - (c + \lambda)(W_0^{(2)} - w_r) = -c(V_0^{(2)} - v_r), & z \in (0, \tau), \\ V_0^{(2)}(0) = v_r, \quad W_0^{(2)}(0) = w_r - A. \end{cases}$$

It is easy to see the existence of the solution of this problem by standard theory for ordinary differential equations.

By putting $\varepsilon = 0$ in (2.5), we formally get

$$\begin{cases} \gamma k(U_0^{(3)})V_0^{(3)}W_0^{(3)} - aU_0^{(3)} = 0, & z \in (\tau, \infty), \\ cV_0^{(3)\prime} + k(U_0^{(3)})V_0^{(3)}W_0^{(3)} = 0, & z \in (\tau, \infty), \\ W_0^{(3)\prime} - (c+\lambda)(W_0^{(3)} - w_r) = -c(V_0^{(3)} - v_r), & z \in (\tau, \infty), \\ V_0^{(3)}(\tau) = V_0^{(2)}(\tau), \quad |W_0^{(3)}(+\infty)| < \infty. \end{cases}$$

Traveling wave solutions are supposed to be bounded. We supposed that $W_0^{(3)}$ satisfies the boundary condition at ∞. Then, by the similar argument above, we have $U_0^{(3)}(z) \equiv 0$, $V_0^{(3)}(z) \equiv V_0^{(2)}(\tau)$, and $W_0^{(3)}(z) \equiv w_r + c(V_0^{(2)}(\tau) - v_r)/(c+\lambda)$.

Next we consider the inner solution at $z = 0, \tau$. At $z = 0$, we introduce the stretched variable $\xi = z/\varepsilon$. Rewrite (2.1) by using ξ and putting $\varepsilon = 0$. Then we formally get

$$\begin{cases} \ddot{\phi}_0 - (c+\lambda')\dot{\phi}_0 + \gamma k(\phi_0)v_r(w_r - A) - a\phi_0 = 0, & \xi \in (-\infty, \infty), \\ \phi_0(-\infty) = 0, \quad \phi_0(\infty) = U_0^{(2)}(0) \ (= h_+(v_r(w_r - A))), \end{cases}$$

where "$\cdot$" denotes the differentiation with respect to ξ. There is $\overline{A}$ such that for any given $0 < A < \overline{A}$, this problem has a solution $\Phi_1(\xi)$ with a wave speed uniquely determined, denoted by $c = c^*(A)$. The constant $\overline{A}$ is given such as the wave speed $c^*(A)$ corresponds to 0 for $A = \overline{A}$. Note that $c^*(A)$ is continuous with respect to A and decreases monotonically.

Before we consider the inner solution at $z = \tau$, we first define $\alpha(c)$ and $\Phi_1(\xi)$. Let $\alpha(c)$ be a positive constant such as the problem

$$\begin{cases} \ddot{\phi} - (c+\lambda')\dot{\phi} + \alpha(c)\gamma k(\phi) - a\phi = 0, \quad \xi \in (-\infty, \infty), \\ \phi_0(-\infty) = h_+(\alpha(c)), \quad \phi_0(\infty) = 0 \end{cases}$$

has a solution $\Phi_1(\xi)$ for each $0 < c < \overline{c}$. We denote the maximum wave speed by $\overline{c}$, i.e., $\overline{c}$ is such a positive constant as this problem does not have a traveling wave solution for $c > \overline{c}$.

Now we introduce the stretched variable $\xi = (z-\tau)/\varepsilon$ and obtain an inner solution at $z = \tau$. We formally obtain

$$\begin{cases} \ddot{\phi}_0 - (c+\lambda')\dot{\phi}_0 + \gamma k(\phi_0)V_0^{(2)}(\tau)W_0^{(2)}(\tau) - a\phi_0 = 0, & \xi \in (-\infty, \infty), \\ \phi_0(-\infty) = U_0^{(2)}(\tau) \ (= h_+(V_0^{(2)}(\tau)W_0^{(2)}(\tau))), \quad \phi_0(\infty) = 0. \end{cases}$$

If $V_0^{(2)}(\tau)W_0^{(2)}(\tau)$ is equal to $\alpha(c)$, this problem has a solution $\phi_0(\xi) = \Phi_2(\xi)$.

We have defined all outer and inner solutions. Recall that the wave speed c must be $c^*(A)$ for the existence of $\Phi_1(\xi)$. Then, substituting $c = c^*(A)$ into the outer and inner solutions, we formally express our traveling wave solution (u, v, w) as

$$(u,v,w) \sim \begin{cases} (\Phi_1(\frac{z}{\varepsilon}), v_r, W_0^{(1)}(z)), & z \in I_1, \\ (U_0^{(2)}(z) + (\Phi_1(\frac{z}{\varepsilon}) - U_0^{(2)}(0)) + (\Phi_2(\frac{z-\tau}{\varepsilon}) - U_0^{(2)}(\tau)), V_0^{(2)}(z), W_0^{(2)}(z)), & z \in I_2, \\ (\Phi_2(\frac{z}{\varepsilon}), V_0^{(2)}(\tau), w_r + \dfrac{c^*(A)(V_0^{(2)}(\tau) - v_r)}{c^*(A) + \lambda}), & z \in I_3. \end{cases}$$

Unfortunately, the function w is not continuous at $z = \tau$ in general. In addition, we do not see that there does exist the function $\Phi_2(\xi)$, that is, $V_0^{(2)}(\tau)W_0^{(2)}(\tau)$ correspond to $\alpha(c)$. To establish these two conditions, we must choose an appropriate pair (A, τ), which is given in the next lemma.

Lemma 1. *There is a pair (A^*, τ^*) such that it satisfies*

$$\begin{cases} (c^*(A) + \lambda)(W_0^{(2)}(\tau) - w_r) = c^*(A)(V_0^{(2)}(\tau) - v_r), \\ V_0^{(2)}(\tau)W_0^{(2)}(\tau) = \alpha(c^*(A)). \end{cases} \tag{2.9}$$

Proof. To prove this lemma, we evaluate the behavior of the solution of a differential equation

$$(2.10)\qquad \begin{cases} c^*(A)v' = -k(h_+(vw))vw, & z > 0, \\ w' - (c^*(A) + \lambda)(w - w_r) = -c^*(A)(v - v_r), & z > 0, \\ v(0) = v_r, \quad w(0) = w_r - A \end{cases}$$

in the v-w phase space. In particular it is important to study the A-dependency of the solution.

We introduce some notations here (see Figure 2). We define a line L and a hyperbolic curve Π by

$$L = \{(v, w) \mid (c^*(A) + \lambda)(w - w_r) = c^*(A)(v - v_r)\}, \quad \Pi = \{(v, w) \mid vw = \alpha(c^*(A))\},$$

respectively. The line L is through (v_r, w_r), while Π is below (v_r, w_r) because of $\alpha(c^*(A)) < v_r w_r$. The slope of L is positive so that L intersects Π at a unique point in $v > 0, w > 0$, denoted by (v_A, w_A). It is obvious that $v_A < v_r$ and $w_A < w_r$. Let Γ be a segment defined by

$$\Gamma = \{(v, w) \in L \cup \Pi \mid v_A < v < v_r\}.$$

In what follows, we show that the solution of (2.10) is through the intersection (v_A, w_A) for some A.

We note that v' is strictly negative for positive v and w, the initial value of (2.10) is below (v_r, w_r) in the phase space. Due to the continuity and monotonicity of $c^*(A)$ with respect to A, $(v_r, w_r - A)$ is beneath L and above Π. Hence the flow of (2.10) must hit Γ at some z for $0 < A < \overline{A}$, denoted by $z^*(A)$. It is easy to see that $z^*(A)$ is uniquely determined. Since the solution of (2.10) continuously depends on the initial value and parameters, $z^*(A)$ is continuous with respect to A.

We finally prove that there is A such that $(v(z^*(A)), w(z^*(A))) = (v_A, w_A)$ for some A. If A is close to 0, the initial value is near $(v_r, w_r) \in L$. Then v decreases more than w for small $z > 0$ so that $(v(z^*(A)), w(z^*(A)))$ must be on L at $z^*(A)$. On the other hand, $c^*(A)$ tends to 0 as $A \to \overline{A}$, and then the slope of L also tends to 0. Since $w_{\overline{A}} = w_r$ is larger than $w_r - \overline{A}$, $(v(z^*(A)), w(z^*(A)))$ must be on Π at $z^*(A)$. From these facts and the continuity of $c^*(A)$ and $z^*(A)$ with respect to A, we can conclude that there is A^* such that $(v(z^*(A^*)), w(z^*(A^*)))$ matches (v_{A^*}, w_{A^*}) by the intermediate value theorem. We put $\tau^* = z^*(A^*)$. □

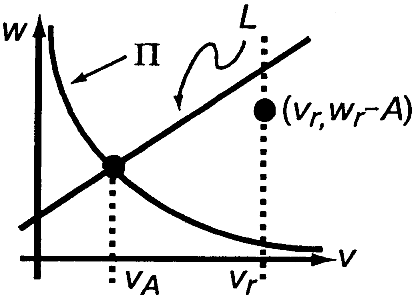

FIGURE 2. The line L and the hyperbolic curve Π in the v-w plane. There is a unique intersection of L and Π, which corresponds to (v_A, w_R).

2.2. The lowest order approximation of (1.11). In this subsection we obtain outer and inner solutions for (1.11) by taking the limit of $\varepsilon \to 0$. When we construct the solutions, we need the relationship between λ and the wave speed c. In the next lemma, we prove that λ must be larger than c.

Lemma 2. *If there is a bounded solution of (1.11) and (1.12), the wave speed c is less than λ.*

Proof. By the second equation of (1.11) and $u \to 0$ as $z \to \infty$, $v(+\infty)$ exists and $v(+\infty) < v_r$. From the third equation of (1.11), we have

$$(\lambda - c)(w_r - w_l) = -c(v_r - v(+\infty)) < 0.$$

Due to $w_r > w_l$, we see $\lambda > c$. □

We first construct outer solutions by the similar argument in the previous section. By putting $\varepsilon = 0$ in (2.6), we have

$$U_0^{(1)}(z) = 0, \quad V_0^{(1)}(z) = v_r, \quad W_0^{(1)}(z) = w_l.$$

By putting $\varepsilon = 0$ in (2.7), we formally get $U_0^{(2)} = h_+(V_0^{(2)}W_0^{(2)})$, and $(V_0^{(2)}, W_0^{(2)})$ is a solution of

$$\begin{cases} cV_0^{(2)\prime} = -k(h_+(V_0^{(2)}W_0^{(2)}))V_0^{(2)}W_0^{(2)}, & z \in (0,\tau), \\ W_0^{(2)\prime} + (\lambda - c)(W_0^{(2)} - w_l) = c(v_r - V_0^{(2)}), & z \in (0,\tau), \\ V_0^{(2)}(0) = v_r, \quad W_0^{(2)}(0) = w_l. \end{cases}$$

Finally, by putting $\varepsilon = 0$ in (2.8), we have

$$\begin{aligned} &U_0^{(3)}(z) = 0, \quad V_0^{(3)}(z) = V_0^{(2)}(\tau), \\ &W_0^{(3)}(z) = \left(w_l - \frac{c}{\lambda - c}(V_0^{(2)}(\tau) - v_r)\right)(1 - e^{-(\lambda-c)(z-\tau)}) - W_0^{(2)}(\tau)e^{-(\lambda-c)(z-\tau)}. \end{aligned}$$

Note that $W_0^{(2)}(\tau) = W_0^{(3)}(\tau)$ holds. From the boundary condition for the function w at ∞, $W_0^{(3)}(+\infty) = w_l - c(V_0^{(2)}(\tau) - v_r)/(\lambda - c)$ must be equal to w_r. However it does not hold true in general. We will find an appropriate value w_l later.

Next we consider the inner solutions at $z = 0$ and $z = \tau$. At $z = 0$, we introduce the stretched variable $\xi = z/\varepsilon$. Rewrite (1.11) by using ξ and putting $\varepsilon = 0$. Then we formally get

$$\text{(2.11)} \qquad \begin{cases} \ddot{\phi}_0 + (\lambda' - c)\dot{\phi}_0 + \gamma k(\phi_0)v_r w_l - a\phi_0 = 0, & \xi \in (-\infty,\infty), \\ \phi_0(-\infty) = 0, \quad \phi_0(\infty) = U_0^{(2)}(0) \ (= h_+(v_r w_l)). \end{cases}$$

This problem has a solution $\Phi_1(\xi)$ with a wave speed $c = c^*(w_l)$ uniquely determined for each $w_l > w_*$, where w_* is given such as $c^*(w_*) = 0$. Since our interest is in traveling wave solutions with a positive wave speed, we naturally assume this condition. In addition we should consider the upper bound for w_l because $c^*(w_l)$ must be smaller than λ from Lemma 2. Hence we suppose that w_l satisfies $w_* < w_l < w^*$, where w^* are defined as follows. The constant w^* is supposed to be w_r in the case of $\lambda > c^*(w_r)$, while in the case of $\lambda \le c^*(w_r)$, it is defined such as $c^*(w^*) = \lambda$. The wave speed $c^*(w_l)$ is continuous and increases monotonically so that w_*, w^* are uniquely determined.

At $z = \tau$, we introduce the stretched variable $\xi = (z - \tau)/\varepsilon$ and formally get

$$\begin{cases} \ddot{\phi}_0 + (\lambda' - c)\dot{\phi}_0 + \gamma k(\phi_0)V_0^{(2)}(\tau)W_0^{(2)}(\tau) - a\phi_0 = 0, & \xi \in (-\infty,\infty), \\ \phi_0(-\infty) = U_0^{(2)}(\tau) \ (= h_+(V_0^{(2)}(\tau)W_0^{(2)}(\tau))) \quad \phi_0(\infty) = 0. \end{cases}$$

If $V_0^{(2)}(\tau)W_0^{(2)}(\tau)$ is equal to $\alpha(c^*(w_l))$ for w_l, this problem has a solution denoted by $\Phi_2(\xi)$, where α was defined in the previous section.

We have already defined all outer and inner solutions of (1.11). Recall that the wave speed c must be $c^*(w_l)$ for the existence of $\Phi_1(\xi)$. Then, substituting $c = c^*(w_l)$ into the outer and inner solutions, we formally express our traveling wave solution (u, v, w) as

$$(u,v,w) \sim \begin{cases} (\Phi_1(\frac{z}{\varepsilon}), v_r, w_l), & z \in I_1, \\ (U_0^{(2)}(z) + (\Phi_1(\frac{z}{\varepsilon}) - U_0^{(2)}(0)) + (\Phi_2(\frac{z-\tau}{\varepsilon}) - U_0^{(2)}(\tau)), V_0^{(2)}(z), W_0^{(2)}(z)), & z \in I_2, \\ (\Phi_2(\frac{z}{\varepsilon}), V_0^{(2)}(\tau), W_0^{(3)}(z)), & z \in I_3. \end{cases}$$

The function w does not satisfy the boundary condition at $z = +\infty$ in general as described previously. In addition, we do not see that there does exist the function $\Phi_2(\xi)$, that is, $V_0^{(2)}(\tau)W_0^{(2)}(\tau)$ corresponds to $\alpha(c^*(w_l))$. To establish these two conditions, we must choose an appropriate pair (w_l, τ), which is given in the next lemma.

Lemma 3. *There is a pair (w_l^*, τ^*) such that it satisfies*

$$\begin{cases} w_l - \dfrac{c^*(w_l)}{\lambda - c^*(w_l)}(V_0^{(2)}(\tau) - v_r) = w_r, \\ \qquad\qquad V_0^{(2)}(\tau)W_0^{(2)}(\tau) = \alpha(c^*(w_l)). \end{cases} \tag{2.12}$$

Proof. We first introduce several notations. Let (v, w) be a solution of

$$\begin{cases} c^*(w_l)v' = -k(h_+(vw))vw, & z > 0, \\ w' + (\lambda - c^*(w_l))(w - w_l) = -c^*(w_l)(v - v_r), & z > 0, \\ v(0) = v_r, \quad w(0) = w_l. \end{cases} \tag{2.13}$$

Define two lines L_1, L_2 and a hyperbolic curve Π by

$$L_1 = \{(v, w) \mid (\lambda - c^*(w_l))(w - w_l) = -c^*(w_l)(v - v_r)\},$$
$$L_2 = \{(v, w) \mid v = v_r - \frac{\lambda - c^*(w_l)}{c^*(w_l)}(w_r - w_l)\},$$
$$\Pi = \{(v, w) \mid vw = \alpha(c^*(w_l))\}.$$

Since the slope of L_1 is negative, L_1 intersects Π at two points. Let $P_{L_1,\Pi}$ be one of the intersections whose component of v in the v-w plane is less than another point. We denote a unique intersection of L_2 and H by $P_{L_2,\Pi}$. The point P_{L_1,L_2} denotes the intersection of L_1 and L_2. We also set $P_3 = (v_r, w_l)$ and $P_4 = (v_r, \alpha(c^*(w_l))/v_r)$, which are on L_1 and Π, respectively. By these notations, we define a set Γ, which consists of segments of L_1, L_2 and Π, by

$$\Gamma = \{(v, w) \mid (v, w) \in L_2 \text{ between } P_{1,2} \text{ and } P_2\} \cup \{(v, w) \mid (v, w) \in \Pi \text{ between } P_2 \text{ and } P_4\}.$$

On the line L_1, $w' \equiv 0$ and $v' < 0$ so that the solution (v, w) of (2.13) must be Γ at some z. Let $z^*(w_l)$ be the first point of z where (v, w) is on Γ. It is obvious that $z^*(w_l)$ depends on w_l continuously.

Actually, the line L_2 is not included in $v > 0$ for w_l close to w_* because of $c^*(w_*) = 0$. Since $(\lambda - c^*(w_l))(w_r - w_l)/c^*(w_l)$ decreases monotonically with respect to w_l, there is uniquely $\tilde{w}_*$ such that

$$\frac{\lambda - c^*(\tilde{w}_*)}{c^*(\tilde{w}_*)}(w_r - \tilde{w}_*) = 0.$$

Clearly, $w_* < \tilde{w}_*$ holds so that we only consider $\tilde{w}_* < w_l < w^*$ in the following.

We see by the same argument as in the proof of Lemma 1 that (v, w) hits $P_{L_2,\Pi}$ for some w_l, which completes the proof of the lemma. If w_l is near $\tilde{w}_*$, the w-component of $P_{L_2,\Pi}$ is large. Then, (v, w) is on Π for $z = z^*(w_l)$. On the other hand, in the case of $w_l = w^*$, the initial value (v_r, w^*) lies on L_2, which implies that (v, w) is on L_2 for w_l near w^* at $z = z^*(w_l)$. Due to the continuity of $z^*(w_l)$ with respect to w_l, there is w_l^* such that $(v(z^*(w_l^*)), w(z^*(w_l^*)))$ is equal to $P_{L_2,\Pi}$. □

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